Iterative Curvelet Thresholding Scheme for Image Recovery

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Abstract: In this paper, we present an optimized sampling strategy based on curvelet for the data acquisition, which leads to better performance by the sparsity-promoting inversion in comparison with random sampling scheme. One of motivation is to provide a new direction of applications of the curvelet transform for data aggregation in image recovery. An analysis of a data aggregation scheme based on curvelet transforms and thresholding iterative scheme for recovering data from aggregated signals is presented.

Key Words: Curvelet Transforms, Iterative Curvelet Thresholding, Sparsity

1. INTRODUCTION

Applications such as Medical Imaging, Genomic Data Analysis, and Remote Surveillance require higher sampling rates to reconstruct the signal with high precision at lower costs. Further, higher sampling rates increase data to be transmitted requiring more bandwidth else will suffer longer delays. To achieve the objectives of transferring the data with lower bandwidth and within acceptable delays, sample compression is employed. However,

compression may cause distortion while reconstructing the signal from decompressed samples. It is a challenging task to develop compression systems with high compression ratio as well as lower loss of information during compression.

Transform coding is a technique of finding a reference basis or frame to make the represented signal sparse or compressible. Sparsity means "representing a signal of length *n* with *k*<< *n* nonzero coefficients" and compressibility states "*signal can be well approximated with only k non-zero coefficients*". Locations and values of only *k* largest coefficients are preserved at the time of creating the sparse representation. This technique has been widely used in JPEG, JPEG2000, MPEG and MP3 standards [1]. In our study, we describe an aggregation scheme using curvelet transform and proposed a reconstruction scheme based on iterative curvelet thresholding .

2. RELATED WORK

Mathematical basis of curvelet transform [2,3,4] was derived extending Unequally Spaced Fast Fourier Transform (USFFT) that obeys a parabolic scaling relation which says that at scale 2^{-j} each element has an envelope aligned along a ridge of length $2^{-j/2}$ and width 2^{-j}. Transforms were successfully used for compression of image data using the proposed transforms. However, the proposed scheme was not very effective in digital coronization due to tiling effect.

Plonka et al. [5] reviewed applications of curvelet transforms for image/video processing, seismic exploration, fluid mechanics, and simulation of partial differential equations and compressive sensing. A multi-resolution geometric analysis technique with curvelet as the basis function to deal with future challenges was also presented.

Results on threshold of number of samples required for exact reconstruction of sampled data were presented by Qaiser et al. [6]. A convex optimization problem was designed for exact recovery of the compressed signal. It is shown that any signal of N can be made out of n spikes with high probability upto the O(n.log N) where N>>n, using ℓ_1 regularization problem [7]. Donoho et al. [8] designed a compressed data acquisition protocol which performed equivalent to directly acquiring just the important information about the signals, in effect, and not acquiring that part of the data that would eventually just be "thrown away" by lossy compression. However, stability of the proposed transform was not established.

3. DATA AGGREGATION USING CURVELET TRANSFORM

Let there be a set of n pixels divided into k clusters with i^{th} cluster having a population of m_i pixels. The CH receives L bits from each Cluster Member (CM) and fuses the received data into a frame of appropriate size. Observations of m_i pixels of each cluster are column matrix of (m_i x 1) i.e. $X = (x_1, x_2, \ldots x_{m_i})^T$.

A compressed version **Y** of **X** is obtained through a measurement matrix Φ, i.e. **Y** = Φ**X**, where Φ is an *m x n* matrix, named as sensing matrix, with *m*<<*n* and generated through same transform. Each element y_i in the vector **Y** is called a random projection, which can be computed as an inner product of the form

$$y_i = \sum_{j=1}^{n} \phi_{ij} x_j \quad (1)$$

CH receives the data x_j from j^{th} member of the cluster and form **X** s.t. $X = (x_1, x_2, \ldots x_{m_i})^T$ which is considered to be compressible. Thus, the process of data aggregation consists of the two steps namely (i) collection of random

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projections **Y** of **X** and (ii) recovering signal **X** from **Y**. Collection of projections needs to be computed at the CH and recovery will be made at the sink.

Curvelet Transform (CT) introduced by Candes & Donoho [6] is defined as function of x at scale 2^{-j} oriented at θ_l positioned at

$x_k^{(jl)} = R_{\theta_l}^j \ (k_1 2^{-j}, k_2 2^{-\frac{j}{2}})$ w. r. t. CH i.e.

$$\varphi_{jlk}(x) = \varphi_j(R\theta_l(x - x_k^{jl})) \qquad (2)$$

Where, R_θ is the angular position.

$$R_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \text{ and } R_\theta^{-1} = R_\theta^T = R_{-\theta}$$

Hence, curvelet coefficient for a pixel set f is the inner product of CT with the data set.

$$C(j,l,k) = <f, \varphi_{jlk}> \quad = \int_{R^2} f(x)\overline{\varphi_{jlk}(x)}\, dx \text{ in}$$

continuous form

$$= \sum_{i=1}^{n} f(x_i)\varphi_{jlk}(x)^T \qquad \text{in discrete form} \qquad (3)$$

So the arbitrary function $f(x_i) \in L^2(R^2)$ can be reconstructed as $f = \sum_{jlk} < f, \varphi_{jlk} >$ at CH. With equality holding with Parseval relation

$$\sum_{jlk} \left|< f, \varphi_{jlk} >\right|^2 = \|f\|_{L^2(R^2)}^2 \ \forall f \in L^2(R^2) \qquad (4)$$

As standard in scientific computation, curvelet coefficients can never be calculated if f is derived in implicit form and these are the rows of matrix representing the linear transformation, known as Riesz representation.

ANALOG DIGITAL OF THE CURVELET COEFFICIENT

In case of data aggregation, the Cartesian curvelets is defined as

$$\Phi_{jlk}(x) = 2^{\frac{3j}{4}}\Phi_j(S_{\theta_l}^T(x - S_{\theta_l}^T b)) \quad \forall b \in (k_1 2^{-j}, k_2 2^{-\frac{j}{2}})$$

$$\text{and } S_{\theta_l} = \begin{pmatrix} 1 & 0 \\ -\tan\theta_l & 1 \end{pmatrix} \qquad (5)$$

So, the analog digital coefficient of the curvelet transform is given by

$$C(j,l,k) = \sum_{i=1}^{n} f(x_i)U_j(S_{\theta_l}^{-1}x)e^{i<b,S_{\theta_l}^{-1}x>})$$

and through a shearing operation, we get

$$C(j,l,k) = \sum f(S_{\theta_l}x_i)U_j(x_i)e^{i<b,x_i>}) \qquad (6)$$

where, $U_j(x_i) = 2^{-\frac{3j}{4}}W(2^{-j}x_i)V(2\lfloor j/2 \rfloor \dfrac{\theta_l}{2\pi})$ with,

$\sum_{j=-\infty}^{\infty} W^2(2^j r x_i) = 1 \quad \forall r \in (0, \frac{3}{4})$ and

$\sum_{l=-\infty}^{\infty} V^2(t - x_i) = 1 \quad \forall t \in (-\frac{1}{2}, \frac{1}{2})$, $U_j(x_i)$ can be calculated recursively.

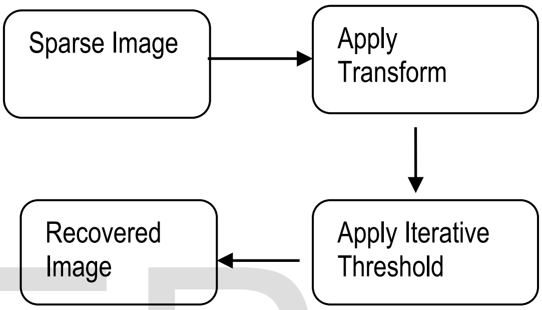
Numerical computation of the equation is straight forward for $\theta_l = 0$ and involves the following 3 steps:

(i) Start with FFT of object f and obtain f (ii) Multiply f with U_j (iii) inverse FT on the range $b = (k_1 2^{-j}, k_2 2^{\frac{j}{2}})$.

But, the inverse Discrete Fourier Transform (DFT) on non-standard sheared grid, $S_{\theta_l}^{-T}(k_1.2^{-j}, k_2 2^{-j/2})$ is very difficult to calculate. So, to recover the data using rectangular grid, we apply the shearing operation to f and re-derive the equation (6) as

$$C(j,l,k) = \sum f(S_{\theta_l}\omega)U_j(\omega)e^{i<b,\omega>}) \qquad (7)$$

Where, ω is obtained by shearing operation on X.



4. IMAGE RECOVERY:

Compressive sensing is a two step process. The first step is to collect the compressed data and in the second phase the data is reconstructed at the fusion centre. Signal recovery is the process of reconstructing the observed signal from the *k-sparse* samples. Researchers [9,10] studied the problem and suggested different techniques for data recovery.

Mathematically, signal reconstruction is equivalent to recovering a sparse signal $V \in R^n$ with V having few non-zero coefficients. All we know about the signal is *m*-non-adaptive measurements X which have been observed from $X = \Phi V$ where, Φ is R^{mxn} matrix (sensing) developed in the previous section from curvelet transform.

If *m* = length of signal, then signals can be reconstructed perfectly. However, for *m*<<*n* several algorithmic approaches have been proposed [11]. The two most appealing algorithms are:

1. Iterative Curvelet Thresholding, and
2. Greedy Pursuit Algorithms (or Orthogonal Matching Pursuit)

Detailed description of the first algorithm is as follows.

ITERATIVE CURVELET THRESHOLDING

Compressive Sensing asserts that we can recover certain signals from far fewer samples or measurements than required in traditional methods. In the measurement

$$Y = AX + \varepsilon \qquad (8)$$

with A_{mxn} measurement matrix *s.t.* $m<<n$ and ε is the measurement noise. Normally, it is an undetermined ill posed problem to recover x from its measurement y. To make the form solvable, CS relies on sparsity.

Algorithm: Sparse Representation – based Classification for Image recovery

Input: A matrix of pixels $\Phi \in R^{mn\,X\,N}$ of C pixels, a pixel $y \in R^N$ and an error e> 0

1. Normalise the columns of Φ to have a L_2 norm of 1
2. Solve the L_1 minimisation problem :

$$\hat{c}_0 = \arg\min_e \|e\|_1 = \left(\sum_{n=1}^{N} |e_n|^2\right)^{\frac{1}{2}}$$

subject to $\|\Phi c - y\|_1 \le \Delta f$

3. Compute the residuals

$$r_i(y) = \left\| y - \Phi_i\,(\hat{c}_0)_i \right\|_2 \quad for\ all\ i \in \{1,2,3,...C\}$$

here Φ_i *and* $(\hat{c}_0)_i$ denote all the pixels and essential coefficients associated with image *i*.

Output *: = identity of y=a rg min* $_i r_i(y)$

$$\hat{L}_y$$

75% pixels removed

a. Original image

Recovery through Iterative curvelet threshholding

b. Reconstructed Image

Sparsity: Let x be the signal data and D is the transform. If x can be decomposed as $x=D\alpha$ where α is the vector coefficient that represent x in D. A signal is said to be sparse if a large number of entries of α are zero or they can be discarded without significant loss of information. In this case, the signal is sparse in the appropriated basis *i.e.* basis of curvelet coefficients.

Considering $y = Rx = RD\alpha$ (9)

R is the sampling matrix, to be vector of n-measurements of the sparse signal x with number of non-zero coefficients $S = \|\alpha\|_0$ $\quad 0 << n << N$ with many more unknowns than observations.

It is concluded that original signal X can be reconstructed from Y with a large probability by solving convex problem $\min\|\alpha\|_2 \;\; s.t. \;\; y = RD\alpha$ (10)

If the number of measurements satisfies

$$n \ge c\mu^2(R,D).S.\log N$$

where, $\mu(R,D) = \sqrt{N}.\max_{1\le i\le n, 1\le j\le n} |< R_i, D_j >|$ and

$$\mu(R,D) \in [1, \sqrt{N}]$$

By solving (10), one can reconstruct sparse t coefficients among all possible α satisfying

$y= RD\alpha$, if the solution coincides with α, then exact recovery is possible

5. CONCLUSIONS

A new method for combining the curvelet transform with iterative thresholding to recover an image is demonstrated and the issue is described as a linear inverse optimal problem using the L_1 norm. Random noise suppression in image data is transformed into an L_1 norm optimization problem based on the curvelet sparsity transform. Compared to the conventional methods such as median filter algorithm, FX deconvolution, and wavelet thresholding, the results of synthetic and field data processing show that the iterative curvelet thresholding proposed in this paper can sufficiently improve signal to noise ratio (SNR) and give higher signal fidelity at the same time. Furthermore, to make better use of the curvelet transform such as multiple scales and multiple directions, we control the curvelet direction of the result after iterative curvelet thresholding to further improve the SNR

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